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Statistical Interpretation of the Planck Constant and the Uncertainty Relation

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A statistically founded derivation of the quanta of energy is presented, which yields the Planck formula for the mean energy of the blackbody radiation without making use of the quantum postulate. The derivation presupposes an ensemble of particles and leads to a statistical interpretation of the Planck constant, which is defined and discussed. By means of the proposed interpretation of h and as an application of it, the quantum uncertainty relation is derived classically and results as a statistical inequality. On the whole this paper is compatible with the statistical ensemble interpretation of quantum mechanics.

1. INTRODUCTION

As is well known, the quantum theory originated from the "quantum postulate," which was introduced by Planck in order to explain in a way compatible with experiment the then open problem of blackbody radiation. Planck postulated that energy is emitted or absorbed only in discrete quantities $n \cdot \varepsilon$, where n ranges over the positive integers; $\varepsilon = h \cdot \nu$ the "quanta of energy"; h represents the Planck constant and ν the frequency of emission (or absorption) of energy. This is the quantum postulate, which created a true revolution in physics and gave us an entirely new conception of the microworld.

Yet Planck himself was never completely satisfied with this new microphysics, which could perhaps have been different if a different solution had been provided to the 1900 blackbody radiation problem. Popper (1967) reminds us that the early quantum-mechanical problems from which the theory was spawned were of a clearly statistical character, adding that *statistical questions demand statistical answers*.

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In this paper we intend to show that an alternative answer exists which leads to a statistical interpretation of the quanta of energy and the Planck constant without affecting the radiation law *per se*. The derivation presented herein indicates that the law could have been derived without the introduction of the quantum—or indeed any—postulate. Also, by means of the statistical interpretation of h, the "uncertainty relation" emerges naturally as a statistical inequality.

Before proceeding to the proposed derivation, we believe it would be of interest to recall in brief the original derivation of the quanta.

2. THE ORIGINAL DERIVATION OF THE QUANTA

The successive attempts of Planck to solve the blackbody radiation problem are clearly reproduced in Jammer (1966). The problem at issue was to find an expression for the mean energy $\langle E\rangle$ of a system consisting of N noninteracting particles (the molecules of the black body), N being very large, at constant temperature T and common frequency ν. Planck considered the thermodynamic relation

$$d\langle S\rangle = d\langle E\rangle/T \tag{1}$$

where $\langle S\rangle$ is the mean entropy, defined by the Boltzmann formula

$$\langle S\rangle = (k\log w)/N \tag{2}$$

with k the Boltzmann constant and w the thermodynamic probabilities (or weights) of the several microstates of the system which correspond to its macrostate.

In order to evaluate w, Planck used an approximative method devised earlier by Boltzmann. Namely, he considered the total energy of the system E_t to consist of n very small individual energy units of measure ε, so that

$$E_t = n\cdot\varepsilon = N\cdot\langle E\rangle \tag{3}$$

where n ranges over the positive integers. Because of (3), w can be calculated from

$$w = \frac{(N+n-1)!}{n!(N-1)!} \simeq \frac{(N+n)!}{n!N!}$$

with n, N very large numbers. Then by use of the Stirling formula

$$\log(x!) = x\log x - x$$

for $x = n/N = \langle E \rangle/\varepsilon$. After introduction in (2) and differentiation, Planck got

$$\frac{d\langle S \rangle}{d\langle E \rangle} = \frac{k}{\varepsilon} \log \frac{1 + (\langle E \rangle/\varepsilon)}{\langle E \rangle/\varepsilon} \tag{4}$$

which by integration gives the known result

$$\langle E \rangle = \varepsilon/e^{\varepsilon/kT} - 1 \tag{5}$$

The last equation had to agree with Wienn's experimental result and ε had to be proportional to the frequency ν. Therefore Planck put

$$\varepsilon = h \cdot \nu \tag{6}$$

According to the Boltzmann approximation that had been used in the introduction of ε in (3), Planck should, after the evaluation of the mean energy (5), consider the limit $\varepsilon \to 0$ or $h \to 0$, in order to find the true value of $\langle E \rangle = f(\nu, T)$. But this limit yields

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{e^{\varepsilon/kT} - 1} = kT$$

This result holds true only for low frequencies; while if h is taken as very small but finite, then the fraction $h\nu/kT$ is considerable for high frequencies, and the relation (5) is in complete agreement with experiment for every value of ν. Then Planck realized that he had to introduce the quantum postulate, and he did so "acting desperately," since "*a theoretical explanation had to be supplied, at all cost, whatever the price*" (Jammer, 1966).

3. QUANTIZATION OF PHASE SPACE WITHOUT THE QUANTUM POSTULATE

In this and the next section we shall show how the same problem can be solved as a classical problem of statistical nature; there will be no need to postulate the quanta. We consider a statistical ensemble of N noninteracting particles at constant temperature T, considered as oscillators of common frequency ν. We ask to find the mean energy of the ensemble.

The state of the ensemble is specified by the probability density

$$\rho(q_1, q_2, \ldots, q_f; p_1, p_2, \ldots, p_f)$$

of the $2f$-dimensional phase space. The meaning of ρ is understood according to the equation

$$\delta N = \rho(q_1,\ldots,q_f;\, p_1,\ldots,p_f)\, \delta q_1 \cdots \delta q_f \delta p_1 \cdots \delta p_f \tag{7}$$

where δN determines the number of particles with positions and momenta lying in a selected elementary finite range in phase space.

If we look at the ensemble macroscopically, for $N \to \infty$ we regard ρ and δN as continuous. In fact, in most cases we can put dN instead of δN and $dq_1 \cdots dp_f$ instead of $\delta q_1 \cdots \delta p_f$ in (7) and have a satisfactory approximation. But, in reality, matter is not continuous, and there are experimental situations where the discrete nature of matter makes itself evident and the approximation to the continuum becomes unsupportable, e.g., at high temperatures or frequencies. In such cases we have to look at our ensemble "microscopically," i.e., take into account the discontinuity of matter.

Therefore here the energy probability density is considered to be of the discrete canonical form

$$\rho(E_n) = \exp(-E_n/kT) \tag{8}$$

where $\rho(E_n)$ is understood as the probability of finding the energy E_n in the elementary range $\delta q_1 \cdots \delta p_f$.

The required mean energy of the ensemble is given by the formula

$$\langle E \rangle = \sum_n E_n \rho(E_n) / \sum_n \rho(E_n) \tag{9}$$

In order to evaluate (9) we proceed as follows: We have to define a measure of the elementary range in phase space. This is accomplished if we quantize phase space by dividing it in properly selected equal cells of very small volume. Then we calculate the energy contained in every cell. Evidently because of the statistical nature of the problem, the so-selected phase-space "quantum" will depend on the density $\rho(q; p)$ and will refer to the ensemble of particles and not to the individual particle. Let us therefore consider the variables

$$\delta q \equiv \delta q_i = q_i - \langle q \rangle \qquad \text{and} \qquad \delta p \equiv \delta p_j = p_j - \langle p \rangle, \qquad i, j = 1, 2, \ldots, f \tag{10}$$

with $\langle q \rangle$, $\langle p \rangle$ the averages of positions and momenta in the ensemble, respectively.

An appropriate quantization of phase space is obtained if we take the volume of the elementary cell to be

$$\sigma = |\langle \delta q \delta p \rangle| = |\mathrm{Cov}(q, p)| \tag{11}$$

where Cov(q, p) is the covariance of positions and momenta in the ensemble. We note that another possible choice of elementary volume could be $\delta V = \Delta q \Delta p$, with

$$\Delta q = \{\langle (q_i - \langle q \rangle)^2 \rangle\}^{1/2} \quad \text{and} \quad \Delta p = \{\langle (p_j - \langle p \rangle)^2 \rangle\}^{1/2} \tag{12}$$

representing, respectively, the standard deviations of positions and momenta. However, the inequality

$$\Delta q \Delta p \geqslant |\mathrm{Cov}(q, p)| \tag{13}$$

states that the covariance is smaller and therefore more appropriate for our purpose. Also the inequality

$$|(q_i - \xi)(p_j - \eta)| \geqslant |\mathrm{Cov}(q, p)| \tag{14}$$

where (ξ, η) is any fixed point, shows that the covariance is statistically the least area in phase space.

The quantity σ, defined in (11), is a constant with dimensions of action depending on the probability density $\rho(q; p)$ and assuming nonzero values whenever the q's and p's possess a joint probability density such that they are statistically dependent and correlated. In fact this is the case here; thus σ is a nonvanishing quantity. (For this and other statistical concepts or results used through this paper, see any standard book on probability or statistics, as, e.g., Papoulis, 1965; Weatherburn, 1968).

The above-mentioned properties of σ justify its selection as the "quantum of phase space."

4. THE ELEMENTARY ENERGY OR "QUANTUM OF ENERGY"

The particles which form a blackbody are molecules of matter performing vibrating motion. Planck considered them to be harmonic oscillators, called by him "resonators." This approach is by no means artificial, because the orbits of vibration are simple closed curves in phase space and can be successfully approximated by the elliptic orbits of the harmonic oscillator, by means of the theory of small oscillations. Namely, we consider the energies of the particles to be of the form

$$E = \frac{p^2}{2m} + \frac{q^2}{2}\left(\frac{d^2V}{dq^2}\right)_{q=0} \tag{15}$$

with

$$\left(\frac{d^2V}{dq^2}\right)_{q=0} = m \cdot \omega^2 > 0 \tag{16}$$

The area of an infinitesimal ring between ellipses with parameters E and $E + dE$ is

$$dq\,dp = 2\pi\,dE/\omega \tag{17}$$

and integration of (17) gives

$$\delta q\,\delta p = \frac{2\pi}{\omega}[E(q, p) - E(\langle q\rangle, \langle p\rangle)] \tag{18}$$

with $\delta q, \delta p$ defined in (10).

Taking the absolute averages of both sides of (18) we obtain for the constant σ

$$\sigma = |\langle \delta q\,\delta p\rangle| = \frac{2\pi}{\omega}|\langle E(q, p)\rangle - E(\langle q\rangle, \langle p\rangle)| = \frac{2\pi}{\omega} \cdot \varepsilon \tag{19}$$

where the constant quantity

$$\varepsilon = |\langle E(q, p)\rangle - E(\langle q\rangle, \langle p\rangle)| \tag{20}$$

equals

$$\varepsilon = \frac{(\Delta p)^2}{2}\left(\frac{\partial^2 E}{\partial p^2}\right)_{\langle q\rangle, \langle p\rangle} + \frac{(\Delta q)^2}{2}\left(\frac{\partial^2 E}{\partial q^2}\right)_{\langle q\rangle, \langle p\rangle} \tag{21}$$

with Δq and Δp the standard deviations of q and p defined in (12), and $(\Delta q)^2, (\Delta p)^2$ their variances. Because of (15) and (16), equation (21) becomes

$$\varepsilon = (\Delta p)^2/2m + m\omega^2(\Delta q)^2/2 \tag{22}$$

The constant temperature of the ensemble or, equivalently, the canonical distribution of energies, implies that the variances $(\Delta q)^2$ and $(\Delta p)^2$ in (22) are very small, and so is the energy ε compared to E. Consequently, we can consider the individual energies E as multiples of the "elementary" energy ε and put

$$E \equiv E_n = n \cdot \varepsilon, \qquad n = 0, 1, 2, \dots \tag{23}$$

Also from (19), (20), and $\omega = 2\pi\nu$, we have for the energy ε

$$\varepsilon = \sigma \cdot \nu \tag{24}$$

Introduction of (23), (24) in the expression of the mean energy (9), yields the result of Planck (5), namely

$$\langle E \rangle = \sigma\nu / e^{\sigma\nu/kT} - 1$$

and the limit $\varepsilon \rightarrow 0$ or $\sigma \rightarrow 0$ corresponds to the continuum and gives the known classical result

$$\lim_{\sigma \rightarrow 0} \langle E \rangle = kT$$

5. STATISTICAL INTERPRETATION OF THE PLANCK CONSTANT: DISCUSSION

Comparison of the derived relation $\varepsilon = \sigma \cdot \nu$ with Planck's $\varepsilon = h \cdot \nu$ suggests that $\sigma = h$ or, at least, $\sigma \simeq h$. It is of interest to note that Planck, in a paper in 1912, interpreted the constant h as "the quantum of phase space" (Jammer, 1966), defined by the relation

$$\iint_{E}^{E+\varepsilon} dq\, dp = h$$

and not as "the quantum of action," because action is a nonconservative physical quantity and the existence of a universal constant of action is not physically justified.

The derivation in the previous sections shows that the quanta of energy and the Planck constant can be derived by means of statistics in a straightforward way such that there is no need to postulate them. Therefore we can assume that the constant h and the energy quantum ε characterize or describe a statistical ensemble of particles and not an individual particle. Especially h can be viewed as a statistical constant, characteristic of the microworld, with the physical significance that of providing a measure of the fundamental "distance" of phase space for microsystems.

This conclusion is compatible with the statistical interpretation of quantum mechanics, especially concerning the arguments raised by Popper (1967) and Ballentine (1970) against the usual "orthodox" interpretation of the Heisenberg relation $\Delta q \Delta p \geqslant h$ as "the uncertainty principle." If we accept the statistical interpretation of h, namely,

$$h \simeq \sigma = |\mathrm{Cov}(q, p)|$$

then the Heisenberg relation coincides essentially with the statistical inequality (13), i.e.,

$$\Delta q \Delta p \geqslant |\mathrm{Cov}(q, p)| \simeq h$$

The constant h is considered as characteristic of microscopic statistical ensembles which can be either space ensembles or time ensembles, as defined by Margenau (1963). When the probability density is such that the positions and momenta are correlated, the constant σ is then a nonvanishing quantity, and this is true in classical statistical mechanics as a rule. The only exception is the extreme classical case of the ideal gas, where the positions are described by the uniform distribution and the marginal probability densities in q and in p are statistically independent (Beck, 1976). Evidently the case $\sigma \to 0$ corresponds to purely "classical" ensembles where the approximation to the continuum applies well, such as in the case of the ideal gas.

The absolute size of σ will naturally depend on the distribution. We may say that a "quantum system" (or, more precisely, a "quantum statistical ensemble") is one for which the constant σ is of the order of magnitude of h.

As an application, we give in the next section a classical derivation of the generalized uncertainty inequality originally proved by Robertson (1929).

6. THE UNCERTAINTY RELATION

Let A and B be two physical quantities, expressed as functions of the positions and momenta not depending explicitly on time, i.e.,

$$A = f(q, p) \qquad \text{and} \qquad B = g(q, p)$$

with f and g assumed well-behaved functions of the variables q and p, defined in a domain D of phase space and functionally independent in D. This implies that their Jacobian determinant is nonzero in D, that is

$$J(q, p) = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial q}\frac{\partial f}{\partial p} = \{A, B\} \neq 0 \tag{25}$$

where $\{A, B\}$ is the Poisson bracket of A and B.

If (25) holds, then to any elementary area in the AB plane corresponds an elementary area in the domain D of phase space, namely,

$$dA\, dB = |\{A, B\}| \cdot dq\, dp \tag{26}$$

Before proceeding, let us consider the position and momentum as two random variables, and the domain D as described by a probability density $\rho(q, p)$ such that q and p are correlated, i.e., $\mathrm{Cov}(q, p) \neq 0$.

The mean values of A and B expressed in terms of $\rho(q, p)$ are by definition

$$\left.\begin{aligned} \langle A \rangle &= \iint_D f(q, p)\rho(q, p)\, dq\, dp = f(q_0, p_0) \\ \langle B \rangle &= \iint_D g(q, p)\rho(q, p)\, dq\, dp = g(q_0, p_0) \end{aligned}\right\} (q_0, p_0) \in D$$

where the integrals are replaced by sums whenever the variables q and p are of the discrete type.

Integration of (26) yields

$$(A - \langle A \rangle)(B - \langle B \rangle) = \int_{q_0}^{q}\int_{p_0}^{p} |\{A, B\}|\, dq\, dp \tag{27}$$

By use of the mean value theorem, we write (27) as

$$(A - \langle A \rangle)(B - \langle B \rangle) = \langle |\{A, B\}| \rangle \cdot (q - q_0)(p - p_0) \tag{28}$$

Taking the absolute averages of (28), we get

$$|\langle (A - \langle A \rangle)(B - \langle B \rangle) \rangle| = \langle |\{A, B\}| \rangle \cdot |\langle (q - q_0)(p - p_0) \rangle| \tag{29}$$

in which the left-hand side is recognized as the covariance of A and B. Hence we put

$$|\langle (A - \langle A \rangle)(B - \langle B \rangle) \rangle| = |\mathrm{Cov}(A, B)| \tag{30}$$

The known inequalities

$$\langle|\{A, B\}|\rangle \geqslant |\langle\{A, B\}\rangle| \tag{31}$$

$$|\langle(q-q_0)(p-p_0)\rangle| \geqslant |\mathrm{Cov}(q, p)| = \sigma \tag{32}$$

$$\Delta A\,\Delta B \geqslant |\mathrm{Cov}(A, B)| \tag{33}$$

and (30), introduced in (29) yield the result

$$\Delta A\,\Delta B \geqslant |\langle\{A, B\}\rangle| \cdot \sigma \tag{34}$$

The above inequality states that the product of the standard deviations of any two canonically conjugate physical quantities $A = f(q, p)$ and $B = g(q, p)$ in terms of the density $\rho(q, p)$, has a lower bound if and only if q and p are correlated. In this case the bound depends on the given probability density. This is a classical result, which is valid also for macroscopic physical systems, as e.g., cars, if we are given their joint distribution in q and in p on, say, weekend vehicular traffic.

In order to have inequality (34) for quantum systems we consider $\sigma = h$ and translate the Poisson bracket into Hilbert space language, by use of the Dirac correspondence rule $\{A, B\} \to (i/h)[A, B]$, as the classical commutator. Then (34) becomes the Robertson generalized "uncertainty" relation, namely,

$$\Delta A\,\Delta B \geqslant |\langle[A, B]\rangle| \tag{35}$$

where now the averages are understood by means of the quantum state function ψ or the corresponding state statistical operator.

The derivation of the uncertainty relation is compatible with the statistical interpretation of quantum mechanics, and is a "statistical dispersion principle" (Ballentine, 1970). The classical probability density $\rho(q, p)$ as well as the quantum state ψ are taken as describing a statistical ensemble of physical systems.

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